Notes on the Emission Line Profile of a Spherically Symmetric, Expanding Shell

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I should not have taken you through the long preamble about multi-variable integrals and Jacobians:

$$\iiint dVol = \iiint dxdydz = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \sin\theta \, dr d\theta d\phi \, .$$

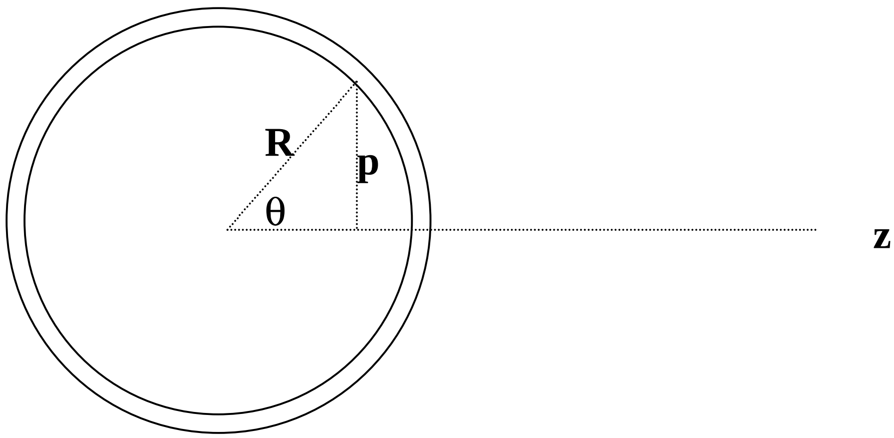
For our problem, this discussion was unnecessary: As we'll see below, the r^2 term doesn't matter for a thin shell at fixed r and, more importantly, the $\sin\theta$ term comes in "naturally" from geometrical considerations. In fact, Geneviève derived it at the board, as I'll show below).

Also, my (temporary) insistence on the additional $\cos\theta$ "projected area" term was incorrect; it is not relevant in this case (but, notice how $\cos\theta$'s keep popping up in the first couple of sections of Rybicki & Lightman).

[Actually, this effect confused me and Stan (who, being a theorist, is much more adept than I at this kind of stuff, and also was a post-doc with George Rybicki at Harvard in the 80's) for a whole afternoon in my office back in October when we were starting to work on this project: If the shell were *totally* thin, then you would need the cosine projection term – you effectively see none of the *side* of a spherical shell or of any infinitely thin surface whose normal is perpendicular to your line-of-sight (i.e. viewing angle).]

But the key to this problem is that we're doing a *volume* integral – we care about the amount of X-rays being emitted from each unit volume of the (thin, but not infinitely thin) shell; This emission is isotropic and not affected by absorption…

So, given these assumptions, let's restate the problem and derivation with a little more care and a lot more clarity.



First, the overall luminosity, L, (photons/sec) is assumed to the same for every volume element in the shell, and is given by:

$$dL = \eta dV,$$

where η is the "volume emissivity", or photons/sec/cm^3. You can put all sorts of physics into this term, such as density dependence. For now, we'll leave it general and constant within the shell. And dV is the differential volume element.

Let's start by writing an expression for the volume of the portion of the shell that is between θ and dθ:

$$dV = 2\pi pRd\theta\Delta R.$$

Here,

2π*p* is the circumference of the ring of constant θ (*p* is what Geneviève called *d* at the blackboard).

***R*d**θ is the arc-length of the ring in the θ-direction (thanks, Eric, for pointing out the need for the *R*).

Δ***R*** is the thickness of the shell (for now, I won't express this as a differential since the shell, while thin, may not be infinitessimally so – and in any case, we won't be integrating over *R*).

So we have:

$$dL = \eta dV = \eta 2\pi pRd\theta\Delta R.$$

Expressing p in terms of θ:

$$p = R\sin\theta.$$

Note that this is essentially where the geometrical factor in the volume integral—the Jacobian—comes in. It basically says that each dφis smaller, the closer one gets to the "poles" (θ=0, 180 degrees). Note that the dφ is hidden in the 2π. We don't need to express it explicitly because of symmetry.

Substituting the last equation into the previous one:

$$dL = \eta dV = \eta 2\pi R^2 \Delta R \sin\theta d\theta \; .$$

Note that the dimensions work out: Volume proportional to $R^2\Delta R$.

Now, we want to know the differential luminosity as a function of wavelength, λ. Our assumption is that each volume element of the shell emits at the same wavelength, λ_O. This wavelength is effectively infinitely narrow, but due to the Doppler shift we can observe a range of wavelengths from $\lambda = -\lambda v_{shell}/c$ to $\lambda = \lambda v_{shell}/c$. Where v_{shell} is, of course, the velocity of the shell, which we're assuming to be constant throughout the shell.

But, the *radial* velocity (i.e. the velocity toward or away from us) is given by:

$$\frac{\lambda}{\lambda_O} = \frac{-v_r}{c} \; ,$$

where λ represents the difference between the observed Doppler shifted wavelength and the emitted, or laboratory, wavelength—usually denoted Δλ, but I want to keep the extraneous notation to a minimum. Note that the negative sign just enforces the convention that negative velocities correspond to blue shifts, while positive velocities correspond to redshifts. Expressing v_{r} in terms of θ:

$$\lambda = \frac{-\lambda_O v_{shell}}{c} \cos\theta \; .$$

Now, differentiating both sides:

$$d\lambda = \frac{-\lambda_O v_{shell}}{c} d(\cos\theta)$$

and noting that

$$d(\cos\theta) = -\sin\theta d\theta$$

we have:

$$d\lambda = \frac{\lambda_O v_{shell}}{c} \sin\theta d\theta$$

or

$$\sin\theta d\theta = d\lambda \frac{c}{\lambda_O v_{shell}} \, .$$

And back to our differential luminosity:

$$dL = \eta \, 2\pi R^2 \Delta R \sin\theta d\theta \, .$$

Substitute in the radial velocity – Doppler shift differential:

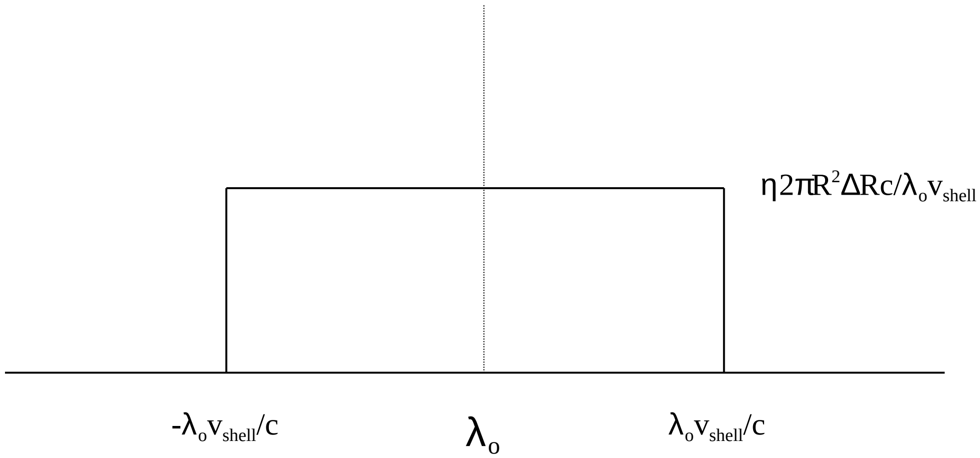
$$dL = \eta \, 2\pi R^2 \Delta R \frac{c}{\lambda_O v_{shell}} d\lambda \, .$$

Expressing this as the differential luminosity per unit wavelength:

$$\frac{dL}{d\lambda} = \eta 2\pi R^2 \Delta R \frac{c}{\lambda_O v_{shell}} = constant \quad \text{on} \quad \frac{-\lambda_O v_{shell}}{c} < \lambda < \frac{\lambda_O v_{shell}}{c}$$

$$= 0 \qquad \text{otherwise.}$$

This is just our line profile:



There is **no** dependence on λ (under our assumptions).

Physically what's happening is that the angular dependence of the size of the rings on the shell ($R\sin\theta$) is combining with the $d\theta$ from the width of the rings to exactly cancel the $\sin\theta d\theta$ dependence of the differential wavelength on the differential radial velocity.

Let's recall: The "shell model" is important because:

1. It's easy to solve
2. Shock waves tend to heat thin regions
3. (Most importantly) You can build more realistic models by summing over many shells (which can have $\eta(r)$).

This type of philosophy is very productive for doing science:

· Keep things simple.

- Always have a physical picture in mind.
- Build up gradually to a realistic model, always keeping complicating details to a minimum (but never forgetting what approximations you're making).

Now, I think we can work our way up to reading the paper I wrote with Stan (which involves nothing more than assuming a velocity structure for the wind, making η a function of radius, and including absorption by the (cold part of the) wind). But first, we should read pp. 1-15 of Rybicki and Lightman:

- It is important to understand flux, intensity, and optical depth
- Work your way up to the "formal solution"
- But don't worry too much about momentum flux, radiation energy density, or radiation pressure.

When you do look at Owocki & Cohen, you'll notice a few things (we'll have to talk about these, and more, of course):

- We dispense with the $\theta - v - \lambda$ mapping and instead use the Dirac delta function to make the correspondence between position in the wind and wavelength in the line.
- Wind absorption is accounted for by a simple exponential term, but it gets somewhat complicated by the fact that *cylindircal* coordinates are the natural system for absorption in a wind, rather than spherical coordinates.
- $\mu = \cos\theta$

We'll discuss more on Friday about stellar winds, velocity laws, etc.