Quantum Coins, Dice and Children: Probability and Quantum Statistics

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We discuss counterintuitive aspects of probabilities for systems of identical particles obeying quantum statistics. Quantum coins and children (two level systems) and quantum dice (many level systems) are used as examples. It is emphasized that, even in the absence of interactions, (anti)symmetrizations of multi-particle wavefunctions destroy statistical independences and often lead to dramatic departures from our intuitive expectations.

One of the most fundamental differences between classical and quantum mechanics is the necessity of introducing quantum statistics — Bose–Einstein statistics [1,2] for particles with integer spins and Fermi–Dirac statistics [3,4] for particles with half-integer spins — for systems of identical particles. As a part of the standard physics curriculum, quantum statistics are usually introduced in courses on statistical physics or quantum mechanics with special emphasis on their applications on statistical systems. For example, Bose–Einstein statistics provide a natural understanding of the blackbody spectrum, and the Fermi theory on electronic bands of condensed matter systems has its foundation on Fermi–Dirac statistics.

On the other hand, probability theories with quantum statistics are rarely discussed. This is a curious omission, given the close connection between classical statistics and probability theory. In this letter, we fill this gap by studying several simple examples on quantum systems of identical particles with incomplete information. The results are often intriguing and counterintuitive. It turns out that, by introducing quantum statistics, statistical independence between measurements on different particles is lost even in the absence of interactions between these particles. These examples can be useful in teaching quantum statistics as they highlight the differences between classical and quantum statistics.

I. QUANTUM COIN TOSSING

We will start with the simplest possible example — the quantum coin tossing problem. (Our quantum coin tossing problem has little to do with another problem with the same name in quantum information theory.) Each quantum coin is a particle in one of the two possible quantum states, labeled "heads" (H) or "tails" (T), which are *a priori* equally likely. It is clear that the probability of getting a "heads" is 50%, regardless of the statistics of the coin. Now consider tossing a set of two coins, in that we mean preparing a mixed state for which all distinct allowable quantum two-particle states are *a priori* equally likely. These conditions are physically realizable for systems with two low-lying single particle discrete levels, which are well separated from the other levels. More specifically, both the energy splitting between the two states, δ, and the interaction energy between particles in these states, ϵ, are much smaller than the temperature, so that by equipartition both states are equally likely to be occupied. The temperature is in turn much smaller than Δ, the energy splitting between these two low-lying states, and the rest of the spectrum, so that these higher states are essentially empty. In other words, the temperature T should be chosen in such a way that $\epsilon, \delta \ll T \ll \Delta$. If such conditions are satisfied, what is the probability the outcome is two "heads"? The answer depends on which statistics the coins obey.

• With classical statistics, *i.e.*, where the particles are distinguishable, there are four possible outcomes:

$$\text{HH}, \qquad \text{HT}, \qquad \text{TH}, \qquad \text{TT}. \tag{1}$$

Since all four outcomes are *a priori* equally likely, the probability for HH is 1/4. This is applicable to tossing macroscopic coins, where quantum effects are negligible.

• With Bose–Einstein statistics, where the allowable states must be symmetric under exchange, there are only three possible outcomes:

$$\text{HH}, \qquad (\text{HT}+\text{TH})/\sqrt{2}, \qquad \text{TT}. \tag{2}$$

Consequently, the probability for HH increases to 1/3. This is applicable, for example, to a simple system of two bosons in an external potential with doubly degenerate ground states labeled as H and T. It is also applicable to two photons in a rectangular optical cavity with dimensions $a \times a \times b$ ($a \gg b$). Such a cavity has two degenerate ground states, which can be labeled as H and T, respectively. Then the probability of finding both photons in the H state is 1/3. (This example has been studied in Dirac's *"The Principle of Quantum Mechanics"* [5].)

• With Fermi–Dirac statistics the outcomes of HH and TT are forbidden as the allowable states must be antisymmetric under exchange; there is only one possible state:

$$(\mathrm{HT}-\mathrm{TH})/\sqrt{2} \tag{3}$$

The probability for HH is obviously zero. This is applicable to a system of two fermions in an external potential with doubly degenerate ground states.

This above analysis clearly shows that the outcomes of measurements on the two coins are not statistically independent. Classically, two systems are usually regarded as statistically independent if they do not interact with each other. This, however, is not necessarily true for quantum mechanical systems of identical particles, where the two-particle wavefunction is entangled unless it can be written as the product of two single-particle wavefunctions. More precisely, for probability applications where one studies mixed states, correlations occur unless the two-particle density matrix can be factored into two density matrices, each describing one of the particles. Bosonic (fermionic) wavefunctions, however, are obtained *via* symmetrization (antisymmetrization) of independent two-particle wavefunctions, and such symmetrizations or antisymmetrizations destroy statistical independence. It is manifestly clear in the case of fermions: the Pauli exclusion principle, decreeing that two identical fermions cannot be in the same state, is incompatible with statistical independence. The analogous effect for bosons is Bose enhancement, which states that bosons are more likely to be found in the same state than statistically independent particles. This simple example of quantum coin tossing illustrates, in a very compelling way, the differences between classical and quantum statistics.

We mention in passing that one can easily generalize the above analysis to the following problem. For n dice, each equally likely to be any of k state (one of which is called "•"), what is the probability that all of them end up being in the "•" state? For distinguishable particles there are k^n distinct possible outcomes, and the probability for any one of them is k^{-n}. For fermions the probability for an "all • state" is trivially 0 (for $n > 1$), and for bosons it is easy to show that there are $\binom{k+n-1}{n}$ distinct possible outcomes. Since these outcomes are all equally likely, the probability for the "all • state" is $1/\binom{k+n-1}{n}$, which is always larger than k^{-n}. In other words, Bose statistics always increases the chance of finding two identical bosons in the same state; Bose enhancement is really an enhancement. (It is important to note that the above analysis holds if and only if there are exactly k accessible states as stated in the problem. The answer will be different if, for example, there are k doublets (*i.e.*, $2k$ accessible states) and one of the doublets is labeled "heads".)

Lastly, a word of caution: real coins and dice do not behave like quantum coins and dice — they are essentially classical objects. Coins and dice are always distinguishable from one another, while the discussion above is only applicable to indistinguishable particles. Even usual quantum systems such as electrons in an external magnetic field do not behave as quantum coins as described above. The analysis above is valid only if there are exactly two allowable states, while an electron in a magnetic field has two spin states *for each accessible spatial quantum state*. There are even more allowable states for real coins and dice, which are distinguished not only by the spatial location but also for physical variations. As a result, the terminologies like "quantum coins" should be taken in a metaphorical sense only.

II. CONDITIONAL PROBABILITIES: THE QUANTUM CRIB

Now we will move on to conditional probabilities, which are even more intriguing and counterintuitive. Consider the following famous problem:

(I) Two children sleep in a crib. If one is chosen at random and turns out to be a boy, what is the probability that both are boys?

(II) Two children sleep in a crib. If at least one of them is a boy, what is the probability that both are boys?

The answer is well known: 1/2 for the question (I), 1/3 for question (II). These answers presume that the two genders are *a priori* equally likely, and also that the children are distinguishable objects and their genders are statistically independent. (Whether these assumptions are strictly true in the real world is beyond the scope of this paper.) But what if we assume the children obey quantum statistics instead? In order to study this question, we will reformulate the above puzzle in the following way to make it applicable to quantum particles:

Consider two identical particles, each being equally probable of being in one of two quantum states: either boy (B) or girl (G) at the same spatial position, with all distinct allowable gender combinations being *a priori* equally likely. (Here the terminologies "boy" and "girl" are used in a metaphorical sense only — real children are distinguishable classical objects. Recall the discussion at the end of the previous section.) We will adopt the following shorthand: "the particle is a B" stands for "the particle is in state B". Then:

(I) One particle is selected in a random manner. If it is a B, what is the probability that the other one is also a B?

(II) Both particles are measured and at least one of them is a B. What is the probability that the other is also a B?

Both of these questions can be easily answered by listing the elements of the spaces of possible combinations. For distinguishable children, the space of possible combinations is {BB, BG, GB, GG}. Out of the four combinations three of them have at least one B, but among them only one is BB, the answer to the question (II) is 1/3, as forecasted above. On the other hand, since all four combinations are equally probable, and for each outcome both particles are equally likely to be selected, there are $4 \times 2 = 8$ equally likely cases:

$$\begin{array}{cccc} \underline{\mathrm{B}}\mathrm{B}, & \underline{\mathrm{B}}\mathrm{G}, & \underline{\mathrm{G}}\mathrm{B}, & \underline{\mathrm{G}}\mathrm{G}, \\ \mathrm{B}\underline{\mathrm{B}}, & \mathrm{B}\underline{\mathrm{G}}, & \mathrm{G}\underline{\mathrm{B}}, & \mathrm{G}\underline{\mathrm{G}}; \end{array} \tag{4}$$

where the underlined particle is being selected. Since in four of these cases B is selected, and among them only two cases the remaining particle is a B, the answer to the question (I) is $2/4 = 1/2$. This answer reflects that the two particles are presumed to be statistically independent, and knowledge of one of the two children does not have any implication for the other child.

The situation is dramatically changed if these children obey quantum statistics instead. It is easy to see that for fermionic children, the BB combination is forbidden by Pauli exclusion principle, and hence the answer to both questions above is: 0. For bosonic children, with the space of possible combination being {BB, (BG+GB)/$\sqrt{2}$, GG}, two out of the three combination have at least one B, and one of them is BB, so the answer to question (II) is 1/2, in contrast to 1/3 for the case with distinguishable children. The analogy of Eq. (4) is

$$\begin{array}{ccc} \underline{\mathrm{B}}\mathrm{B}, & (\underline{\mathrm{B}}\mathrm{G}+\mathrm{G}\underline{\mathrm{B}})/\sqrt{2}, & \underline{\mathrm{G}}\mathrm{G}, \\ \mathrm{B}\underline{\mathrm{B}}, & (\mathrm{B}\underline{\mathrm{G}}+\underline{\mathrm{G}}\mathrm{B})/\sqrt{2}, & \mathrm{G}\underline{\mathrm{G}}. \end{array} \tag{5}$$

In three of these cases B is selected, and since in two of them the remaining particle is also B, the answer to question (I) is 2/3, not 1/2. Again, we see that Bose statistics enhances the probability of finding two identical bosons in the same state.

In the above, we have analyzed the problems by listing all the possible combinations. This becomes less practical for more complicated problems, and one may wonder if it is possible to re-analyze these problems in a way which can be generalized to more complex settings. Since we are studying mixed states, a natural description is *via* density matrices. Both question (I) and (II) will be re-analyzed in the appendix by using the density matrices formalism. However, the remainder of this paper (except the appendix) is in fact accessible without reference to the density matrices formalism.

III. CONDITIONAL PROBABILITIES: THE QUANTUM DAY CARE CENTER

We will now move on from the crib to the quantum day care center. Consider the following problem:

Consider n quantum children (where $n \gg 1$) in a day care center, where by the equal opportunity laws all distinct allowable gender combinations are *a priori* equally likely. We will define R as the ratio of quantum boys to the total number of children in the day care center. Then

(III) What is the probability distribution of R?

(IV) One child is selected in random, which is found to be a boy. What is the probability distribution of R for the remaining children?

For distinguishable children obeying classical statistics, statistical independence implies that the outcome of the remaining $n-1$ children are not affected by the outcome of the first child. As a result, the probability distribution of R is a sharply peaked Gaussian around $R = 1/2$ for both questions (III) and (IV). On the other hand, one can study this problem for bosonic children by enumeration. There are $n+1$ distinct allowable gender combinations:

$$C_k = \{k \text{ boys}, n-k \text{ girls}\}, \qquad 0 \le k \le n, \tag{6}$$

with all of these combinations *a priori* equally likely, *i.e.*, $P(k) = 1/(n+1)$. As a result, the probability for $R = k/n$ is $P(R = k/n) = 1/(n+1)$ where k is an integer between 0 and n and hence $0 \le R \le 1$. When $n \to \infty$, this approaches the uniform probability distribution:

$$f_0(R) \equiv \frac{dP(R)}{dR} = 1, \qquad \langle R\rangle_0 \equiv \int Rf_0(R)dR = 1/2. \tag{7}$$

Question (IV) asks for the probability distribution of R on the condition that the first child selected is a boy. After the selection, there are only $n-1$ quantum children remaining in the quantum day care center, and hence the number of boys left can be any integer between 0 and $n-1$. Now the probability is $P(k) = 1/(n+1)$ for each gender combination C_k, which after one boy is selected is left with $k-1$ left in the quantum day care center, so by Bayes' formula one has

$$\begin{aligned}
\tilde{P}(m) &\equiv P(m \text{ boys left}|\text{first child selected is a boy}) \\
&= P(m+1 \text{ boys before selection}|\text{first child selected is a boy}) \\
&= \frac{P(\text{first child selected is a boy}|m+1 \text{ boys before selection}) \cdot P(m+1 \text{ boys before selection})}{\sum_{j=0}^{n} P(\text{first child selected is a boy}|j \text{ boys before selection}) \cdot P(j \text{ boys before selection})} \\
&= \frac{(m+1)/n \times 1/(n+1)}{\sum_{j=0}^{n} j/n \times 1/(n+1)} = \frac{2(m+1)}{n(n+1)}.
\end{aligned} \tag{8}$$

(We will give a brief description of Bayes' formula for readers who are not familiar with probability theory. Let H_j ($j = 1, \ldots, N$) be N mutually exclusive events, with probabilities $P(H_j)$. Then $P(H_k|A)$, the conditional probability of a particular H_k upon the condition that another event A occurs, is given by the Bayes' formula:

$$P(H_k|A) = \frac{P(H_k) \cdot P(A|H_k)}{\sum_j P(H_j) \cdot P(A|H_j)}. \tag{9}$$

Discussions of Bayes formula can be found in most standard textbooks on probability theory. See, for example, Fraser [6] or Roe [7].)

Returning to Eq. (8), one can easily check that the probabilities of different possible outcomes add up to unity.

$$\sum_{m=0}^{n-1} \tilde{P}(m) = \sum_{m=0}^{n-1} \frac{2(m+1)}{n(n+1)} = 1. \tag{10}$$

The conditional expectation value of m is

$$\sum_{m=0}^{n-1} mP(m) = \sum_{m=0}^{n-1} \frac{2m(m+1)}{n(n+1)} = \frac{2}{3}(n-1). \tag{11}$$

Since there are $n-1$ quantum children remaining in the quantum day care center, $R = m/(n-1)$, and the conditional expectation value of R is $2/3$, *i.e.*, we expect two-thirds of the remaining quantum children to be boys, having determined that a single child (out of a huge day care center) is male! A little quantum knowledge goes a long way in this problem.

As the number of children in the quantum day care center tends to infinity, *i.e.*, $n \to \infty$, the conditional probability $\tilde{P}$ approaches a linear probability distribution:

$$f_1(R) \equiv \frac{d\tilde{P}(R)}{dR} = 2R, \qquad \langle R\rangle_1 \equiv \int Rf_1(R)dR = 2/3, \tag{12}$$

in agreement with the conditional expectation value obtained above.

As a last example, we generalize the previous case to the quantum die rolling problem:

(V) A quantum die is a quantum mechanical particle, *a priori* equally likely to be in one of k possible states. (Again, the terminology "quantum die" is used in a metaphorical sense only — real dice are distinguishable classical objects.) Consider tossing a set of n quantum dice (where $n \gg 1$), by which we mean preparing a mixed state for which all distinct allowable quantum n particle states are *a priori* equally likely. Let's label one of the states "state 1" and define R to be the fraction of quantum dice being in state 1. Then n' coins are selected in random and N_1 of them turn out to be in state 1, N_2 of them in state 2, *etc.*, such that $N_1 + N_2 + \ldots + N_k = n' \ll n$. What is the probability distribution of R for the remaining dice?

This is a straightforward generalization of questions (III) and (IV), which are recovered by setting $k = 2$, $N_2 = 0$ and $N_1 = 0$ in question (III) or $N_1 = 1$ in question (IV). For distinguishable dice, by statistical independence, the probability distribution of R is sharply peaked around $1/k$. We will show in the appendix, by using the density matrix formalism, that the conditional probability distribution of R for the remaining bosonic dice is

$$f_{N_1,N_2,\ldots,N_k}(R) = R^{\nu_1-1} \cdot (1-R)^{\nu_2+\ldots+\nu_k-1}/B(\nu_1, \nu_2+\ldots+\nu_k), \tag{13}$$

where $\nu_j = N_j + 1$ and $B(x,y)$ is the Beta function, and the conditional expectation value of R is

$$\langle R\rangle_{N_1,N_2,\ldots,N_k} \equiv \int R f_{N_1,N_2,\ldots,N_k}(R)dR = \nu_1/(\nu_1+\nu_2+\ldots+\nu_k). \tag{14}$$

IV. DISCUSSIONS

We emphasize that the examples above are not merely academic but may be experimentally realizable and testable. For example, a Bose–Einstein condensate of F-spin-1 atoms (F-spin is the total spin of the atom, which is the quantum mechanical sum of the total angular momentum of the electron system and the nuclear spin) provides a natural realization of a system of quantum dice with $k = 3$, where the three states correspond to $F_z = 1$,0 and -1 along some axis $\hat{z}$. All distinct allowable combinations of F-spins are *a priori* equally likely as long as the system is isotropic, or alternatively the temperature is sufficiently high that the anisotropic term in the Hamiltonian is negligible, while at the same time being low enough to support a Bose–Einstein condensate. If such a scenario is realizable, a randomly extracted atom from the condensate is equally likely to be in any of the three spin states. If the first atom turns out to be in the $F_z = 1$ state, however, Eq. (14) (with $k = 3$ and $(N_1, N_2, N_3) = (1, 0, 0)$) predicts that half of the remaining atoms will also be in the $F_z = 1$ state.

In our discussion, we have referred to the particles as “coins” (with states heads and tail), “children” (with states boy and girl) and “dice” (with states labeled by dots). It must be understood that these terminologies are being used in a merely metaphorical sense. Real children do not spontaneously fluctuate between boy states and girls states. Macroscopic coins and dices are always distinguishable from one another, both by physical variations and by their locations in space. Quantum statistics applies only to particles that are indistinguishable and sharing the same physical location.

We have seen that counterintuitive results often arise when one tries to study probabilities for systems with identical particles obeying quantum statistics. Given the simplicity of our examples, one may wonder why they are not discussed or even mentioned in most undergraduate textbooks on quantum physics or statistical mechanics. We have attempted a literature search for similar discussions; as far as we know, there is no mention of these topics in most standard textbooks on quantum mechanics and/or statistical physics. On the other hand, as mentioned before, the quantum coin tossing problem with two coins was discussed by Dirac in Ref. [5]. There are also discussions in Griffiths [8] and Stowe [9] which share the philosophy of this paper; the specific examples being considered, however, are different from the ones discussed here. In particular, none of these discussions studied conditional probabilities, which give the clearest and the most counterintuitive manifestation of the differences between classical and quantum statistics.

Returning to the question of why these issues not being brought up in most undergraduate textbooks, the reason, we believe, lies in the observation that these particularly counterintuitive results occur only in systems with a finite number of accessible levels. This condition is rarely met in important physical systems; as a result, these subtleties are seldom discussed in most undergraduate textbooks, which understandably tend to focus on systems with more immediate applications. However, we believe our examples can highlight the differences between classical and quantum statistics, and deserve some discussion in undergraduate classrooms.

The main lesson of this discussion is the lack of statistical independence between identical particles in quantum statistics. For the outcomes of measurements on two particles to be statistically independent, the wavefunctions of the two particles must be disentangled. In other words, the density matrix describing the two-particle mixed state must be factorizable into two separate density matrices, each describing one of the particles. For identical particles obeying quantum statistics, however, their density matrices are always entangled due to (anti)symmetrizations. As a result, the outcomes of measurements on identical particles are always correlated, violating statistical independence.

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APPENDIX

In this appendix, we will re-analyze questions (I) – (IV) in the density matrices formalism, which carries the advantages of being a systematic procedure and can be easily generalized to systems of arbitrary number of particles and accessible states.

Let us start with a single quantum coin, with the density matrix:

$$\rho = \tfrac{1}{2}|H\rangle\langle H| + \tfrac{1}{2}|T\rangle\langle T|, \tag{15}$$

and for two quantum coins obeying classical statistics (*i.e.*, being distinguishable), the two-particle density matrix is

$$\rho_{cl} \equiv \rho \otimes \rho = \tfrac{1}{4}|HH\rangle\langle HH| + \tfrac{1}{4}|HT\rangle\langle HT| + \tfrac{1}{4}|TH\rangle\langle TH| + \tfrac{1}{4}|TT\rangle\langle TT|, \tag{16}$$

and the coefficient $\frac{1}{4}$ of the $|HH\rangle\langle HH|$ gives the probability of getting two "heads" when a set of two coins are tossed. Notice that statistical independence is manifest as the two-particle density matrix ρ_{cl} is the product of two single-particle density matrices ρ. On the other hand, for quantum coins obeying bosonic (fermionic) statistics, the two-particle density matrix is obtained by ρ_{cl} by (anti)symmetrization.

$$\begin{aligned} \rho_{BE} &\equiv \lambda_{BE}\ S\ \rho_{cl}\ S = \tfrac{1}{3}|HH\rangle\langle HH| + \tfrac{1}{3}|S\rangle\langle S| + \tfrac{1}{3}|TT\rangle\langle TT| \\ \rho_{FD} &\equiv \lambda_{FD}\ A\ \rho_{cl}\ A = |A\rangle\langle A|\ ; \end{aligned} \tag{17}$$

where S and A are the symmetrization and antisymmetrization projection operators, respectively; the λ's are normalization constants to ensure that the density matrices are properly normalized, *i.e.*, $\mathrm{Tr}\rho_{BE} = \mathrm{Tr}\rho_{FD} = 1$, and

$$|S\rangle = (|HT\rangle + |TH\rangle)/\sqrt{2}, \qquad |A\rangle = (|HT\rangle - |TH\rangle)/\sqrt{2}. \tag{18}$$

Again, the probabilities of getting two "heads" can be read off as the coefficients of the operator $|HH\rangle\langle HH|$. The coefficients are 1/3 and 0 for bosonic and fermionic statistics, respectively, confirming the values obtained through listing. After all, the density matrix formalism is simply a systematic way to generate and organize the list of all possible combinations, with the probability of each combination appearing as the coefficient of the respective projection operator.

As for the problems on conditional probabilities, the condition decreed in question (II), namely at least one of the quantum children is a boy, can be imposed by projecting out the subspace of "all girls" by the projection operator $P = \mathbf{1} - |GG\rangle\langle GG|$, with $\mathbf{1}$ denoting the identity operator. Acting P onto the two-particle density matrices ρ_{BE} and ρ_{FD} above (and renaming "H" as "B" and "T" as "G"), the projected density matrices are,

$$\begin{aligned} \bar{\rho}_{BE} &\equiv \bar{\lambda}_{BE}\ P\ \rho_{BE}\ P = \tfrac{1}{2}|BB\rangle\langle BB| + \tfrac{1}{2}|S\rangle\langle S|, \\ \bar{\rho}_{FD} &\equiv \bar{\lambda}_{FD}\ P\ \rho_{FD}\ P = |A\rangle\langle A|, \end{aligned} \tag{19}$$

where the $\bar{\lambda}$'s are again normalization constants. Again, the conditional probabilities of BB can be easily read off.

The condition decreed in question (I), that a randomly chosen quantum child turns out to be a boy, can be imposed by using the "boy annihilation operator" a_B, satisfying

$$a_B|m_B \text{ boys}, m_G \text{ girls}\rangle = \sqrt{m_B}|m_B - 1 \text{ boys}, m_G \text{ girls}\rangle. \tag{20}$$

After randomly taking a child out of the crib and find that it is a boy, the density matrix of the remaining child is

$$\begin{aligned} \tilde{\rho}_{BE} &\equiv \tilde{\lambda}_{BE}\ a_B\ \rho_{BE}\ a_B^{\dagger} = \tfrac{2}{3}|B\rangle\langle B| + \tfrac{1}{3}|G\rangle\langle G|, \\ \tilde{\rho}_{FD} &\equiv \tilde{\lambda}_{FD}\ a_B\ \rho_{FD}\ a_B^{\dagger} = |G\rangle\langle G|, \end{aligned} \tag{21}$$

where as before $\tilde{\lambda}$'s are normalization constants. The conditional probabilities of the remaining quantum child is a boy (so that both quantum children are boys) again appear as coefficients.

In the remainder of this appendix, we derive the answers (7) and (12) to the quantum day care center problem for bosonic children. Instead of tackling questions (III) and (IV) specifically, we will study the more general question (V). Questions (III) and (IV) are recovered by setting $k = 2$, $N_2 = 0$ and $N_1 = 0$ in question (III) or $N_1 = 1$ in question (IV).

Recall that the "state 1 annihilation operator", a_1, annihilates a quantum die in state 1, and one can analogously define a_j for other states, with $1 \leq j \leq k$. For any *complex* unit vector $\vec{r} = (r_1, \ldots, r_k)$ satisfying $\sum_{j=1}^{k} |r_j|^2 = 1$, the linear combination $A_{\vec{r}} = \vec{r} \cdot \vec{a}$ (where $\vec{a} = (a_1, \ldots, a_k)$) satisfies $[A_{\vec{r}}, A_{\vec{r}}^\dagger] = 1$ (with $\hbar$ also set to unity), and is the annihilation operator for a quantum die in a particular state described by the "polarization vector", $\vec{r}$. It is convenient to parameterize the components of $\vec{r}$ in the following way.

$$\begin{aligned}
r_1 &= e^{i\alpha_1}\,\cos\theta_1, \\
r_2 &= e^{i\alpha_2}\,\sin\theta_2\,\cos\theta_2, \\
\vdots \quad & \quad\quad \vdots \\
r_{k-1} &= e^{i\alpha_{k-1}}\,\sin\theta_1\,\sin\theta_2\ldots\sin\theta_{k-2}\,\cos\theta_{k-1}, \\
r_k &= e^{i\alpha_k}\,\sin\theta_1\,\sin\theta_2\ldots\sin\theta_{k-2}\,\sin\theta_{k-1}.
\end{aligned} \tag{22}$$

Note that, while a *real* unit vector in $\mathbf{R}^k$ lies on a $k-1$-dimensional sphere and hence is described by $k-1$ angles θ_j, a *complex* unit vector in $\mathbf{C}^k$ lies on a $2k-1$-dimensional sphere and k extra phases α_j are needed.

The density matrix of a state with n atoms ($n \gg 1$), all polarized in the $\vec{r}$ direction, is given by,

$$\rho_{\vec{r}} = (1/n!)\,(A_{\vec{r}}^\dagger)^n|0\rangle\langle 0|(A_{\vec{r}})^n. \tag{23}$$

However, as stated in the problem, all distinct allowable combinations are equally likely. As a result, the density matrix ρ for such a state will be a superposition of $\rho_{\vec{r}}$ for all $\vec{r}$.

$$\rho_0 = \frac{\Gamma(k/2)k!}{2\pi^{3k/2}n!}\int dr_1\,dr_1^*\ldots dr_k\,dr_k^*\,\delta(|\vec{r}|-1)\,(A_{\vec{r}}^\dagger)^n|0\rangle\langle 0|(A_{\vec{r}})^n, \tag{24}$$

where the asterisks represent complex conjugation and $\Gamma(k/2)k!/(2\pi^{3k/2}n!)$ is an overall normalization factor to ensure that $\mathrm{Tr}\,\rho_0 = 1$. Since the angles α_j are the phases of r_j, this density matrix can be rewritten as

$$\rho_0 = \frac{2^k\Gamma(k/2)}{2\pi^{k/2}}\frac{k!}{n!}\int_0^\infty |r_1|\,d|r_1|\ldots|r_k|\,d|r_k|\int_0^{2\pi}\frac{d\alpha_1}{2\pi}\ldots\frac{d\alpha_k}{2\pi}\,\delta(|\vec{r}|-1)\,(A_{\vec{r}}^\dagger)^n|0\rangle\langle 0|(A_{\vec{r}})^n. \tag{25}$$

Then we can express the real unit vector $(|r_1|, \ldots, |r_k|)$ in terms of the angles θ_j. Integrating over the Dirac delta distribution $\delta(|\vec{r}|-1)$ gives a factor of $2\pi^{k/2}/(2^k\Gamma(k/2))$, and ρ_0 can be recast as

$$\rho_0 = \frac{1}{n!}\int d\Omega_{\mathbf{C}}\int_0^{2\pi}\frac{d\alpha_1}{2\pi}\ldots\frac{d\alpha_k}{2\pi}\,(A_{\vec{r}}^\dagger)^n|0\rangle\langle 0|(A_{\vec{r}})^n = \int d\Omega_{\mathbf{C}}\int_0^{2\pi}\frac{d\alpha_1}{2\pi}\ldots\frac{d\alpha_k}{2\pi}\,\rho_{\vec{r}}, \tag{26}$$

where

$$d\Omega_{\mathbf{C}} \equiv \prod_{j=1}^{k-1} dP^{(j)} \equiv \prod_{j=1}^{k-1}(k-j+1)\sin^{k-j}\theta_j\cos\theta_j\,d\theta_j, \qquad \int_{\theta_j=0}^{\theta=\pi/2} dP^{(j)} = 1. \tag{27}$$

The interpretation of Eq. (26) is clear. The measure $d\Omega_{\mathbf{C}}$ gives the probability distribution $f^{(j)}(\theta_j)$ in the domain $[0, \pi/2]$.

$$f^{(j)}(\theta_j)\,d\theta_j \equiv (k-j+1)\sin^{k-j}\theta_j\cos\theta_j, \qquad \int_0^{\pi/2} f^{(j)}(\theta_j)d\theta_j = 1. \tag{28}$$

On the other hand, the phases α_j are equally likely to take any value between 0 and 2π.

$$f(\alpha_j) = 1/2\pi. \tag{29}$$

Note that the probability distribution of all three angles θ_j and the phases α_j are independent of each other.

We are interested in evaluating the expectations of the number operators n_1 under the density matrix (26). It is convenient to introduce the observable $R = n_B/n$, denoting the fraction of quantum dice in state 1. Notice that this definition of R coincides with that in problem (III). Since $n_1 = n\cos^2\theta_1$, the observable R can be re-expressed in terms of the angles θ_1 as $R = \cos^2\theta_1$. It is straightforward to rewrite $dP^{(1)}(\theta_1)$ in terms of R.

$$d\Omega_{\mathbf{C}} = f_{0,\ldots,0}(R)dR, \qquad f_{0,\ldots,0}(R) = (1-R)^{k-2}/(k-1), \tag{30}$$

where the subscripts remind us that no die of any state has been removed. In particular, for $k = 2$, we have the answer to question (III): the probability distribution of R is given by $f_0(R) = 1$.

Now, with the distribution function $f_{0,\ldots,0}(R)$, it is straightforward to solve problem (V), which is to evaluate the conditional probability distribution given that n' coins have been removed, and among them N_j of them are found to be in state j. The density matrix after the selection, which can be written as $\rho_{N_1,N_2,\ldots,N_k} = \lambda\ (a_1^{N_1} a_2^{N_2} \ldots a_k^{N_k})\, \rho_0\, (a_1^{N_1} a_2^{N_2} \ldots a_k^{N_k})^\dagger$, where λ is a normalization constant and ρ_0 is the density matrix defined in Eq. (26). Since $\rho_1 \neq \rho_0$, the *conditional* probability distribution of R is no longer given by $f_{0,\ldots,0}(R)$, but instead

$$\begin{aligned} f_{N_1,N_2,\ldots N_k}(R) &= \frac{R^{N_1} \cdot (1-R)^{N_2+\ldots+N_k} f_{0,\ldots,0}(R)}{\int_0^1 R^{N_1} \cdot (1-R)^{N_2+\ldots+N_k} f_{0,\ldots,0}(R) dR} \\ &= \frac{R^{\nu_1-1} \cdot (1-R)^{\nu_2+\ldots+\nu_k-1}}{B(\nu_1, \nu_2+\ldots+\nu_k)}, \end{aligned} \tag{31}$$

reproducing Eq. (13) with $\nu_j = N_j + 1$. In particular, with $k = 2$ and $(N_1, N_2) = (1, 0)$, question (V) reduces to question (IV) with the answer

$$f_1(R) = 2R, \tag{32}$$

which agrees with Eq. (12).

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